11^{TH} European Mathematical Cup

10^{th} December 2022 - 18^{th} December 2022

Junior Category

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Marking schemes

Problem 1.

Solution

- Part A (2 points): Proving that all n which are divisible by 60 satisfy the condition.
- Part B (8 points in total): Proving that any n which satisfies the condition is divisible by 60.
  - B1-P (1 point): Proving n is even.
  - B1 (3 points): Proving n is divisible by 4.
  - B2 (2 points): Proving n is divisible by 3.
  - B3 (3 points): Proving n is divisible by 5.

Notes on marking:

- If a contestant proves that for some $a > 1$, all numbers of the form $60 \cdot a \cdot k$ for $k \in \mathbb{N}$ are solutions, they should be awarded **1 point** out of possible **2 points** for Part A.
- Points B1 and B1-P are not additive.

Problem 2.

First solution.

- Part A (7 points in total): Proving $x - y = 1$.
  - A1 (4 points): Proving $(x-y)^2$ is a positive integer.
  - A2 (3 points): Proving $x - y = 1$ by bounding $(x-y)(x+y)$.
- Part B (3 points in total): Finishing the problem.
  - B1 (1 point): Concluding that $x = \frac{a+1}{2}$, $y = \frac{a-1}{2}$ for some positive integer $a > 1$.
  - B2 (2 points): Proving that only odd a satisfy the conditions.

Second solution.

- Part A (7 points in total): Proving $x - y = 1$.
  - A1 (4 points in total): Proving $(x-y)^2$ is a positive integer.
    - A1-1 (2 points): Writing x and y as solutions $\frac{a \pm \sqrt{D}}{2}$ to a quadratic equation.
    - A1-2 (2 points): Establishing $x - y = \sqrt{D}$.
  - A2 (3 points): Proving $x - y = 1$ by bounding $(x-y)(x+y)$.
- Part B (3 points): Finishing the problem.
  - B1 (1 point): Concluding that $x = \frac{a+1}{2}$, $y = \frac{a-1}{2}$ for some positive integer $a > 1$.
  - B2 (2 points): Proving that the conditions are only satisfied for odd a.

Third solution.

- Part A (7 points in total): Proving $x - y = 1$.
  - A1 (3 points in total): Proving the bounds on $x - y, x$.
    - A1-1 (1 point): Proving $1 \leqslant x - y < 1 + \frac{1}{a}$.
    - A1-2 (2 points): Proving the bounds on x.
  - A2 (4 points in total) Using that xy is an integer to prove $x - y = 1$.
    - A2-1 (2 points) Bounding the difference between $\frac{a^2-1}{4}$ and xy.
    - A2-2 (2 points) Proving that a must be odd.
- Part B (3 points): Finishing the problem by considering the case where a is odd.

Fourth solution.

- Part A (7 points in total): Proving $x - y = 1$.
  - A1 (4 points): Proving $(x-y)^2$ is an integer.
  - A2 (3 points in total):
    - A2-1 (1 point): Proving $1 \leqslant \sqrt{D} < 1 + \frac{1}{x+y}$.
    - A2-2 (2 points): Using AM-GM inequality to prove $\sqrt{D} \leqslant \frac{4}{3}$.
- Part B (3 points): Finishing the problem.
  - B1 (1 point): Concluding that $x = \frac{a+1}{2}$, $y = \frac{a-1}{2}$ for some positive integer $a > 1$.
  - B2 (2 points): Proving that only odd a satisfy the conditions.

Notes on marking:

- In all four solutions, in Part B, minor flaws should result in a **1 point** deduction.
- If a contestant has not solved Part A (up to minor flaws), they can score at most **2 points** in Part B.
- Stating that all pairs of the form $(n+1, n)$ are a solution is worth at most **1 point** for Part B. This is not additive with the remaining points for Part B, but is additive with Part A.
- Points from different solutions are not additive.

Problem 3.

First Solution.

- Part A (6 points in total): Considering the homothety at C which sends H to M.
  - A1 (1 point): Explaining that the homothety sends T to F.
  - A2 (1 point): Explaining that the homothety sends E to J.
  - A3 (4 points): Proving that CJ is tangent to the circumcircle of MFJ.
- Part B (4 points in total): Angle-chase and conclusion.
  - B1 (1 point): Using tangent-chord to obtain $\angle CJM = \pi - \angle MFJ$.
  - B2 (3 points): Final angle-chase and conclusion.

Second Solution:

- Part A (6 points in total): Proving the similarity of triangles CJF and CDH.
  - A1 (1 point): Using the symmetry around the bisector of $\angle ACB$ to obtain equalities $\angle JCF = \angle DCF$, $CD = CE$, $CM = CF$.
  - A2 (1 point): Obtaining $\frac{CJ}{CF} = \frac{CD}{CH}$.
  - A3 (4 points): Concluding via side-angle-side theorem.
- Part B (4 points in total): Angle-chase and conclusion.
  - B1 (1 point): Using tangent-chord to conclude $\angle HDC = \angle CJF = \angle EFC$.
  - B2 (3 points): Final angle-chase and conclusion.

Notes on marking:

- In both solutions, in part B2, a contestant which hasn't made the previous steps (up to minor flaws) can be awarded at most **1 point** out **3 points** available for that part.

Problem 4.

First solution.

- Part A (9 points in total): Proving that there is no lovely family with more than 8 sets.
  - A1 (6 points in total): Showing that there exist intersecting sets A, B such that their union is [300], by contradiction.
    - A1-1 (1 point) Showing that there is no chain of length 3 in F.
    - A1-2 (1 point) Showing that any set in F can have at most two subsets in F.
    - A1-3 (2 points) Using other members of F, showing that in fact any pair of sets in F must be disjoint.
    - A1-4 (2 points) Showing that F can not contain four pairwise disjoint sets, thus obtaining a contradiction.
  - Part A2 (3 points): Using the existence of such sets to show that $|F| \leqslant 8$.
- Part B (1 point): Providing an example of a lovely family with 8 sets.

Second solution.

- Part A (9 points in total): Proving that there is no lovely family with more than 8 sets.
  - A1 (1 point): Explaining that it suffices to focus only on sets which all contain the element $\{1\}$.
  - A2 (1 point): Rephrasing the problem (and the given condition) in terms of tables of zeroes and ones.
  - A3 (3 points): Proving the Lemma.
  - A4 (2 points): Proving the existence of three distinct rows which contain a single one.
  - A5 (2 points): Explaining why this is a contradiction with the problem's statement.
- Part B (1 point): Providing an example of a lovely family with 8 sets.

Notes on marking:

- In the Second Solution, if one proves Part A3 but without rephrasing in terms of tables (Part A2), they should receive the point for A2 as well.
- In the Second Solution, there are different ways to rephrase the Lemma from Part A3. Any equivalent (or almost equivalent) statement from which the rest of the solution can be deduced should still be worth **3 points**.
- Points from different solutions are not additive.