Mathproblems
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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the grade and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: `mathproblems-ks@hotmail.com`.

Solutions to the problems in this issue should arrive before
October 25, 2023

Problems

152. *Proposed by Marian Dinca, Bucharest, Romania and Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Rumania.*
Let a, b, c and d be the lenghts of the sides of a convex quadrilateral inscribed in a circle with radius R. Prove the inequality

$$\frac{a^2}{b+c+d-a}+\frac{b^2}{a+c+d-b}+\frac{c^2}{a+b+d-c}+\frac{d^2}{a+b+c-d}\geq 2\sqrt{2}R.$$

153. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*
Let $\alpha>1$ and let $a, b\in\mathbb{R}$, $b\neq 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix} 1-\frac{a}{n^\alpha} & -\frac{b}{n} \\ \frac{b}{n} & 1+\frac{a}{n^\alpha}\end{pmatrix}^n.$$

154. *Proposed by Anastasios Kotronis, Athens, Greece and Haroun Meghaichi, student, the University of Science and Technology, Houari Boumediene, Algiers, Algeria.*
Let m be a positive integer and let

$$S_{n,m}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-m}.$$

Show that:

(i) $S_{n,1} = -\ln n - \gamma - \frac{1}{2n} + \mathcal{O}(n^{-2})$ as $n \to +\infty$;

(ii) $S_{n,2} = -\dfrac{\ln^2 n}{2} - \gamma \ln n - \dfrac{\gamma^2}{2} - \dfrac{\pi^2}{12} - \dfrac{\ln n}{2n} + \dfrac{1-\gamma}{2n} + \mathcal{O}\left(n^{-2}\ln n\right)$ as $n \to +\infty$;

(iii) There exist real numbers $a_m, \dots, a_0$ and $b_{m-1}, \dots, b_0$ such that

$$S_{n,m} = \sum_{k=0}^{m} a_{m-k} \ln^{m-k} n + \sum_{k=0}^{m-1} b_{m-k-1} \frac{\ln^{m-k-1} n}{n} + \mathcal{O}\left(n^{-2}\ln^{m-1} n\right)$$

as $n \to +\infty$, and determine them.

155. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Let a be a positive real number, let $(L_n)_{n\geq 0}$ be Lucas sequence and let $(a_n)_{n\geq 0}$ be a positive real sequence such that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{n^2 a_n} = a$. Find

$$\lim_{n\to\infty} \left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n L_n}{(2n-1)!!}} \right).$$

156. *Proposed by Dorlir Ahmeti, University of Prishtina, Republic of Kosovo and Alexander Gunning, Australia.*

With $(2n-1)^3$ identical unit cubes we build a bigger cube. We say any unit cube which is still in the big cube is *removable* if at least three faces of the unit cube are not shared with any other unit cube. We begin by removing a unit cube (by doing this we will cause other unit cubes to become removable), and we may repeat this procedure, removing further unit cubes. What is the minimum number of moves required to remove the unit cube which is in the centre of the big cube?

157. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.*

Prove that

$$2\sum_{n=1}^{\infty} \left(\zeta(3)\zeta(6) - H_n^{(3)} H_n^{(6)}\right) + 7\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^6} = 10\zeta(3)\zeta(5) - 2\zeta(3)\zeta(6) - \frac{23}{12}\zeta(8).$$

where $H_n^{(m)} = 1 + \dfrac{1}{2^m} + \cdots + \dfrac{1}{n^m}$ denotes the nth harmonic number.

158. *Proposed by Sava Grozdev, VUZF University of Finance, Business and Entrepreneurship, Bulgaria, Hiroshi Okumura, Department of Mathematics, Yamato University, Osaka, Japan and Deko Dekov, Stara Zagora, Bulgaria.*

Let ABC be a triangle with side lengths $BC = a, CA = b$ and $AB = c$. Prove that the pedal triangle of the inverse of the orthocenter of the triangle ABC in the circumcircle of the triangle ABC is similar to the orthic triangle of the triangle ABC. Find the similitude ratio as function of a, b, c.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems that have appeared in Math Contests around the world and that are most appropriate for undergraduate Math Olympiad training. Proposals are always welcome. The source of the proposals will appear when the solutions are published.

Proposals

105. Let $f : [0;+\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\,\mathrm{d}x = L.$$

106. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that f has infinitely many zeros, but there is no $x \in (a,b)$ with $f(x) = f'(x) = 0$.

(a) Prove that $f(a)f(b) = 0$.

(b) Give an example of such a function on $[0,1]$.

107. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \le \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup\limits_{\|x\|\le 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

108. Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \cdots = c_{n-2}c_0 = 0$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

109. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0\le x\le 1} |p(x)| > \frac{1}{e^n}.$$

MATHNOTES SECTION

Call For papers

Authors are invited to submit articles for publication. The manuscript must be original and written in English, and can be submitted to any member of the editorial board, as well as mathproblems-ks@hotmail.com. All submitted papers will undergo a reviewing process.

JUNIOR PROBLEMS

Solutions to the problems in this issue should arrive before October 25, 2023

Proposals

66. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.*
Let x, y, z be real numbers in the interval $\left[\frac{1}{2}, 2\right]$. Find the minimum and maximum possible value of

$$f(x,y,z) = \frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1}.$$

67. *Proposed by Daniel Sitaru, Mathematics Department, Colegiul National Economic Theodor Costescu, Drobeta Turnu - Severin, Mehedinti, Romania.*
Let $n \in \mathbb{N}$ such that $n \geq 2$. Prove that in any triangle ABC the following inequality holds:

$$\sum \left(\frac{\sqrt[n]{b} + \sqrt[n]{c} - 2\sqrt[n]{a}}{\sqrt[n]{b} + \sqrt[n]{c}} \right)^2 + \frac{3}{\sqrt[n]{abc}} \prod (\sqrt[n]{b} + \sqrt[n]{c} - \sqrt[n]{a}) \leq 3.$$

68. *Proposed by Michael Rozenberg, Tel Aviv, Israel and Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.*
Let a, b, c and d be non-negative real numbers, no three of which are all 0, and such that $a+b+c+d=4$. Prove that

$$\frac{a^2+b^2+c^2+d^2}{ab+bc+cd+da+ac+bd} + \frac{12abcd}{(ab+bc+cd+da+ac+bd)^2} \geq 1.$$

When does equality occur?

69. *Proposed by Mohammed Aassila, Strasbourg, France.*
Let N be a positive fixed integer. Determine the number of integers $1 \leq n \leq N$ such that

$$11 \cdot 2^{n-1} \equiv 4n + 6 \pmod{13}.$$

70. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.*
The following numbers are written on a board along a straight line, as shown.

$$1 \qquad \tfrac{1}{2} \qquad \tfrac{1}{3} \quad \cdots \quad \tfrac{1}{n-1} \qquad \tfrac{1}{n}$$

Now we add the k-th number with the $(k+1)$-st for each $k = 1, 2, \ldots, n-1$, and write the sum down below in the middle of the two numbers. As such, we create a new line with $n-1$ new numbers and we repeat the same procedure with new line and keep going until we are left with only one number.

For example, if $n = 3$ then:

$$\begin{array}{ccccc} 1 & & \frac{1}{2} & & \frac{1}{3} \\ & \frac{3}{2} & & \frac{5}{6} & \\ & & \frac{7}{3} & & \end{array}$$

Find the last number which is written on the board.